THE MINIMAL OF CONCURRENT ERLANG DISTRIBUTIONS

Reza Pulungan
Dependable Systems and Software
Saarland University

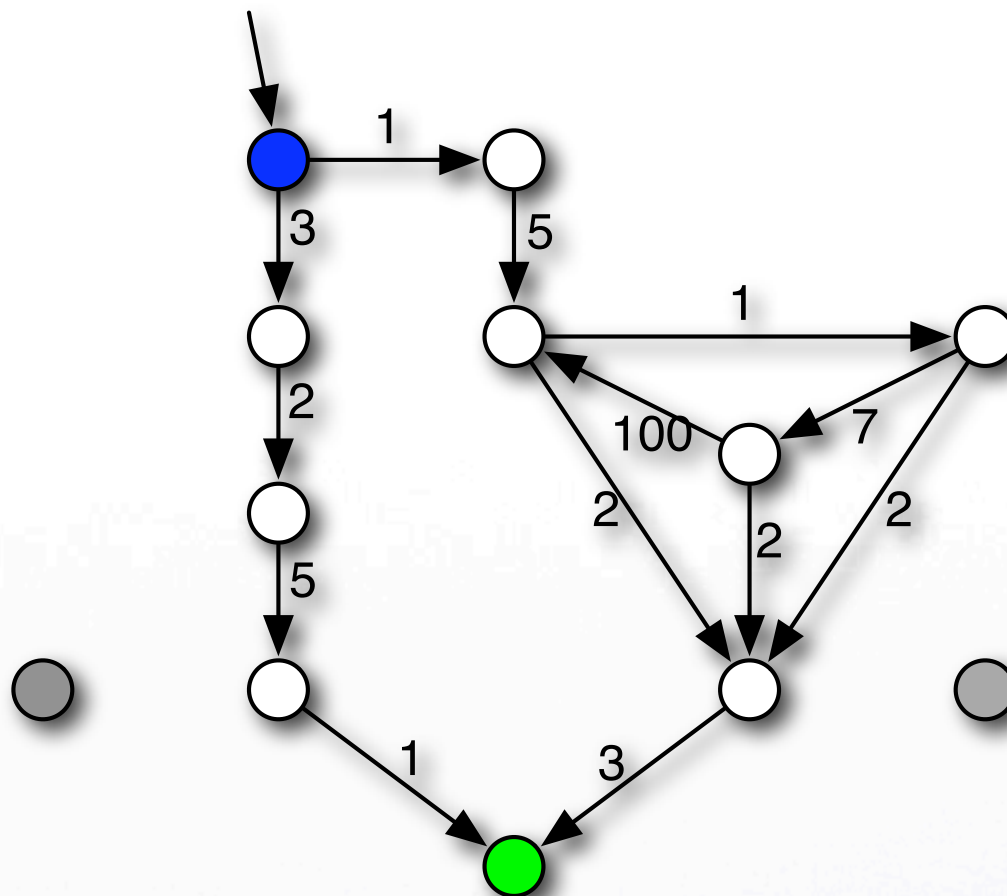
Lorentz Workshop, Leiden, the Netherlands
November 14, 2007

Phase-Type (PH) Distributions

$$\boldsymbol{\pi}(\mathbf{0}) = [\boldsymbol{\alpha}, \alpha_{n+1}]$$

n transient states absorbing state

$$\mathbf{Q} = \begin{bmatrix} \mathbf{A} & -\mathbf{A}\boldsymbol{e} \\ \mathbf{0} & 0 \end{bmatrix}$$



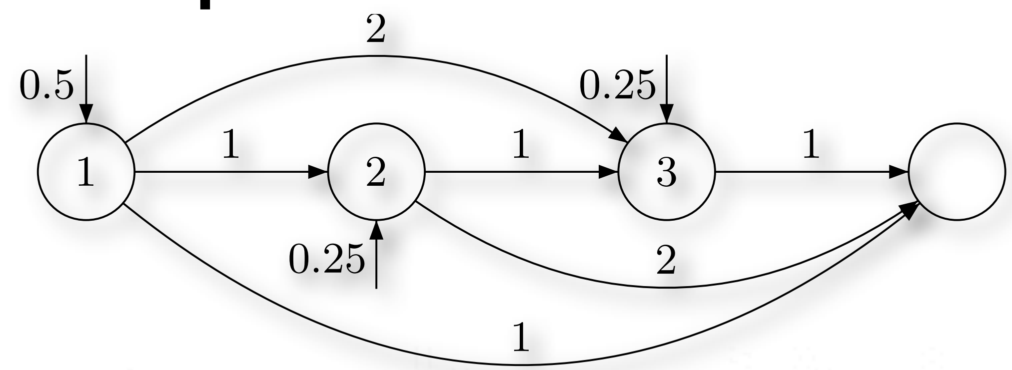
$(\boldsymbol{\alpha}, \mathbf{A})$ is called the representation of the PH.

Acyclic phase-type representation.

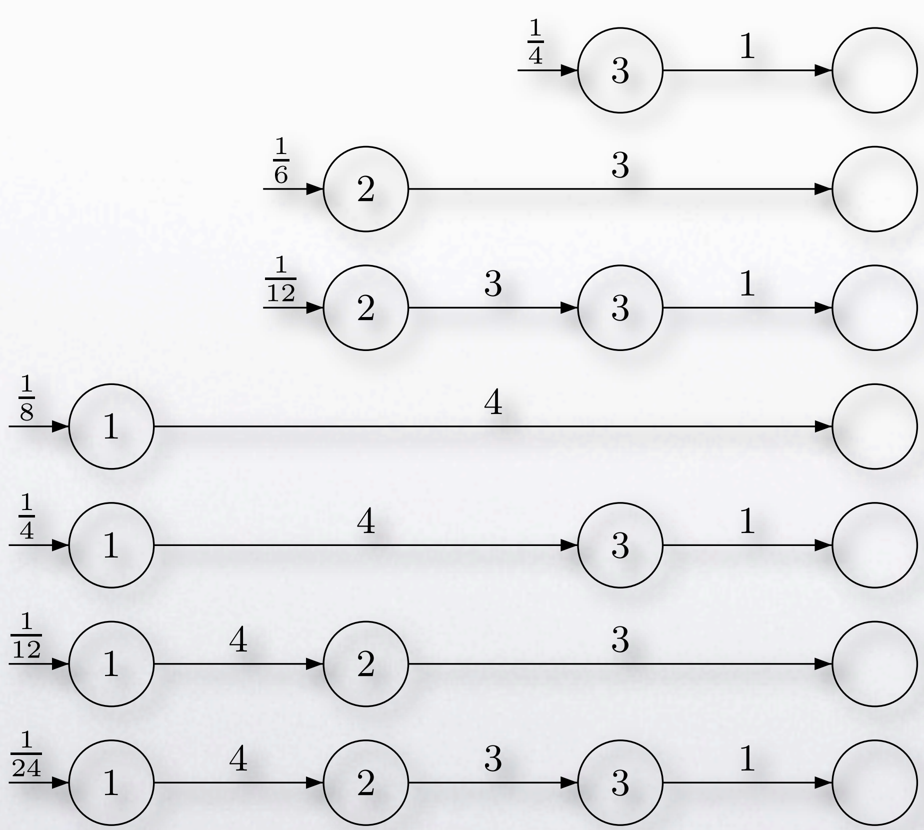
Minimal representation.

Ordered Bidiagonal Representation

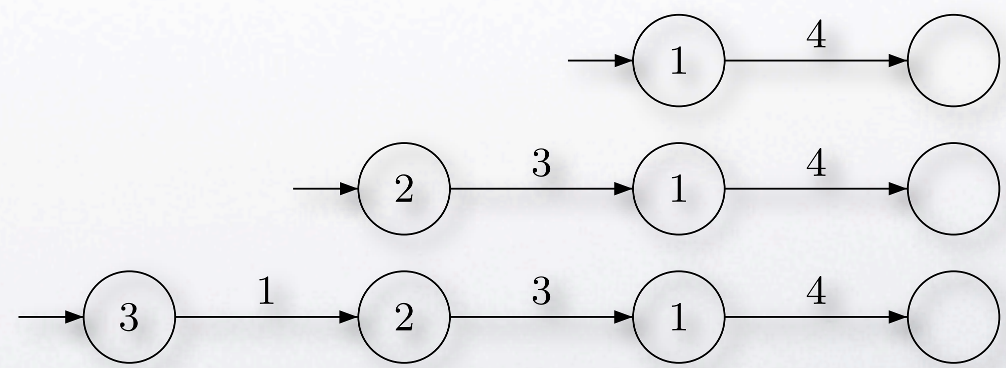
Acyclic Phase-Type Representation



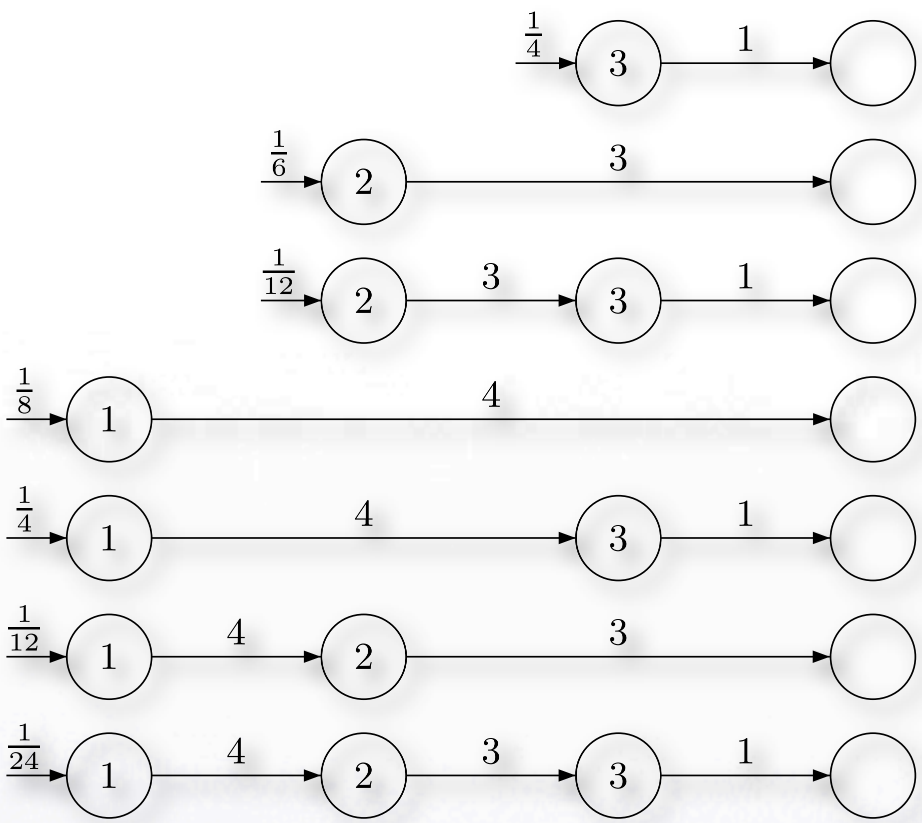
Elementary Series:

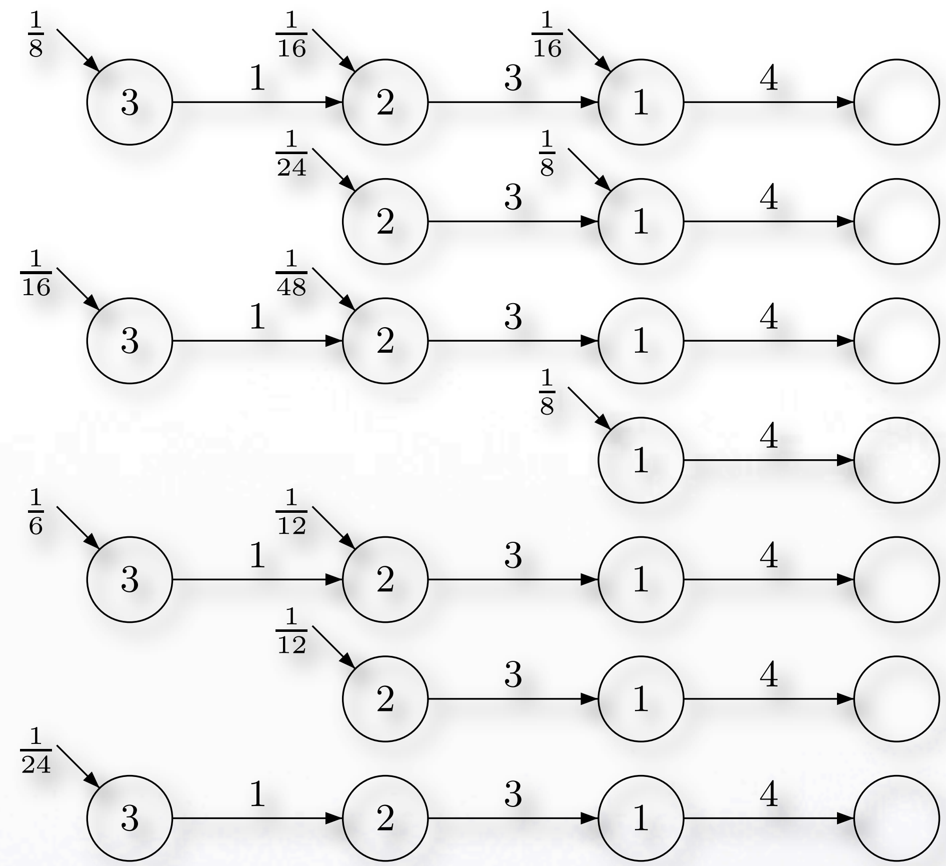


Basic Series:

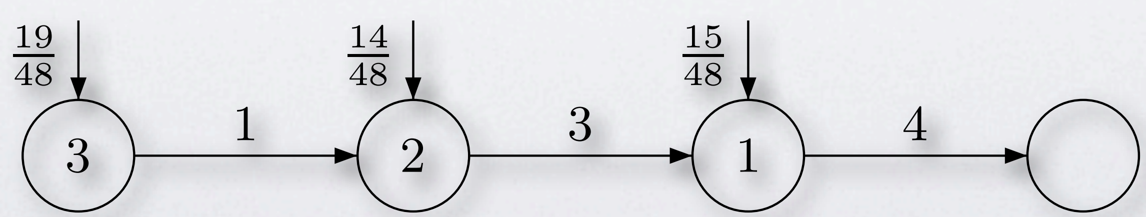


Ordered Bidiagonal Representation





Elementary Series as combinations of Basic Series



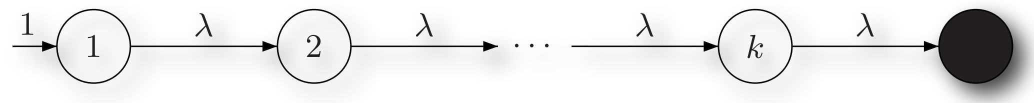
Ordered Bidiagonal Representation

Definition: The core series of the representation can be obtained by removing all excessive multiplicity from each of the basic series.

Lemma: Each of the elementary series of an acyclic phase-type representation is a mixture of the core series of the representation.

Erlang Distributions

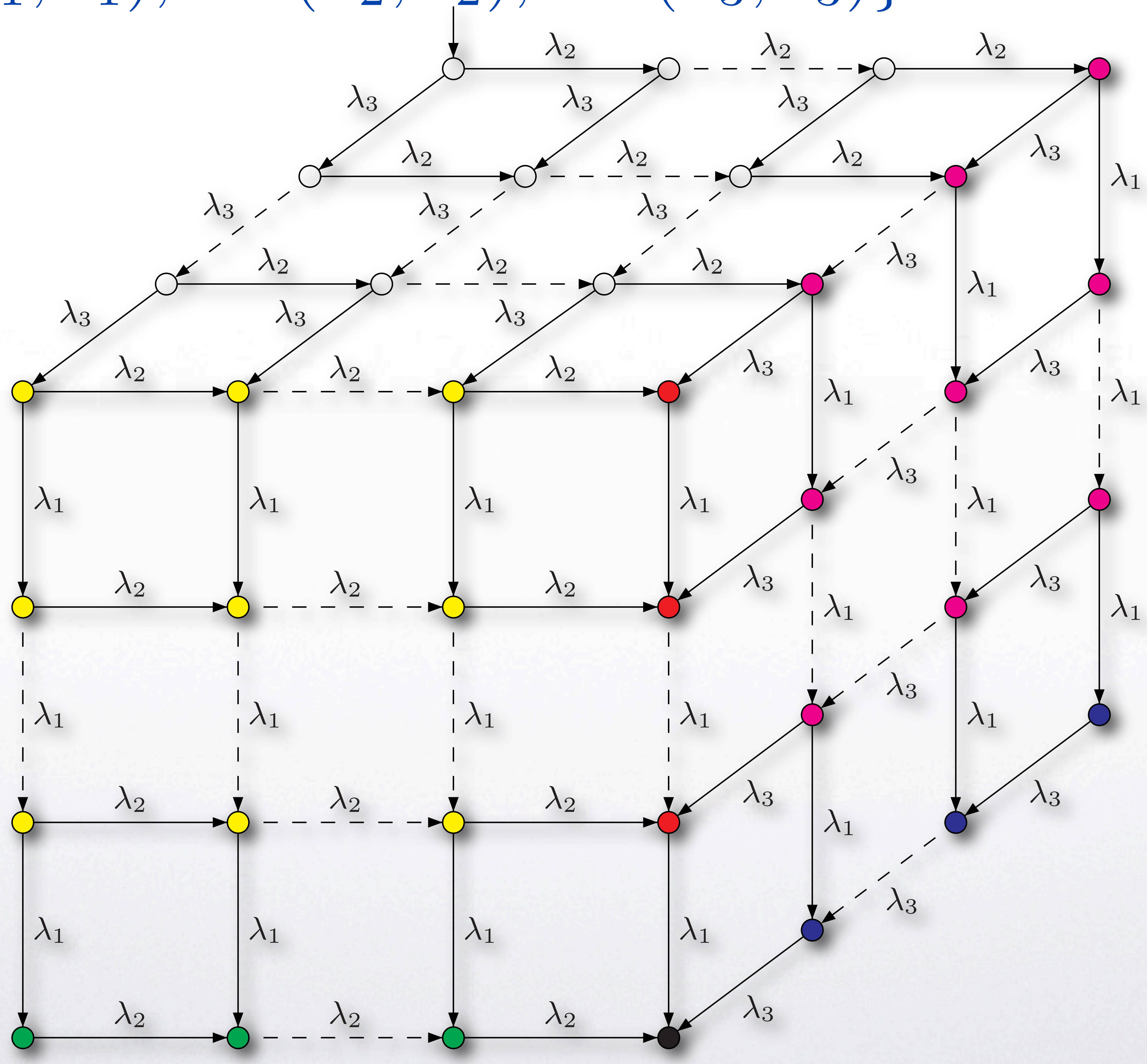
Representation:

$Erl(\lambda, k)$ 

Concurrent Erlangs is their cross product.

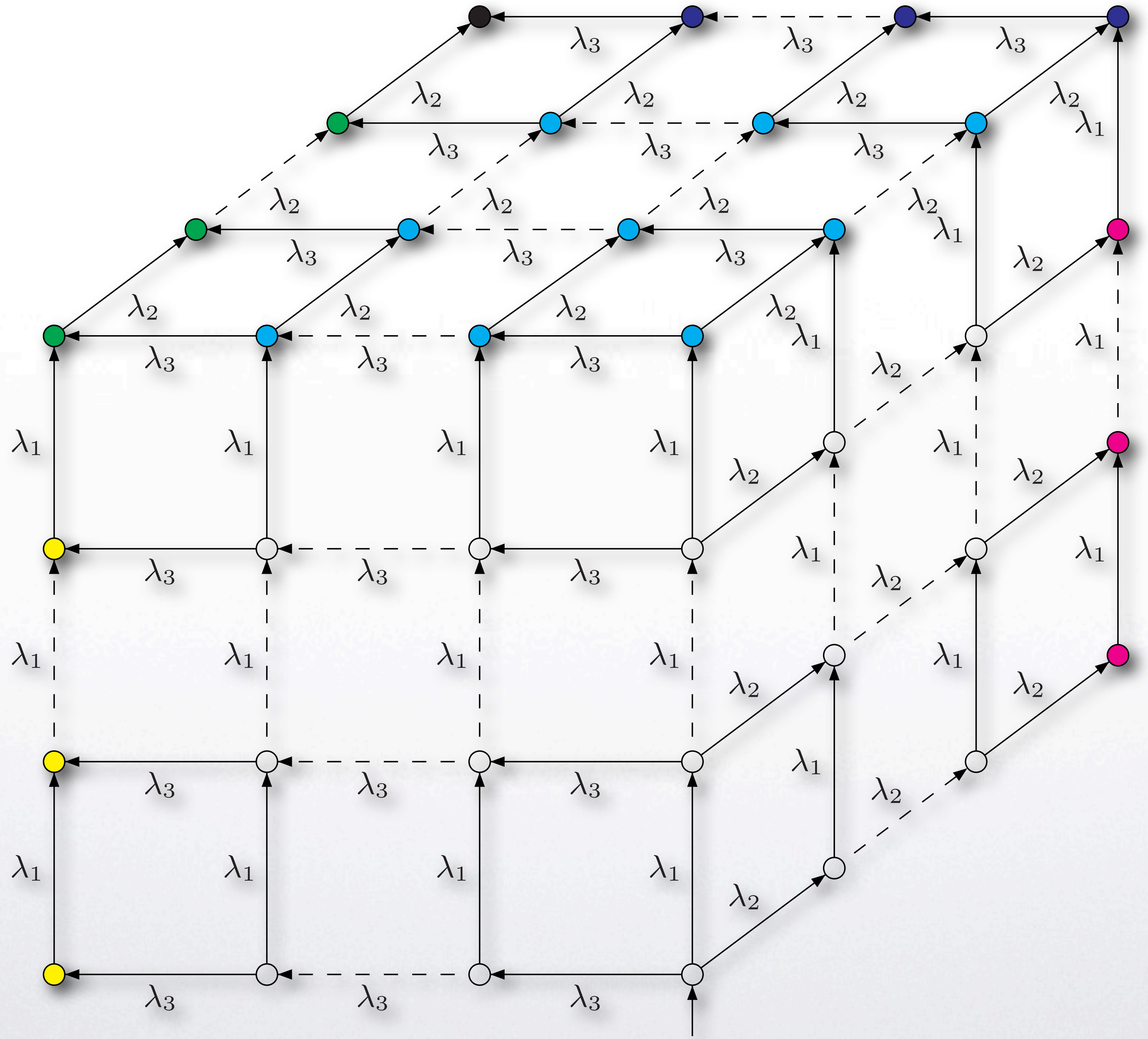
Concurrent Erlangs

$\max\{Erl(\lambda_1, k_1), Erl(\lambda_2, k_2), Erl(\lambda_3, k_3)\}$



Concurrent Erlangs

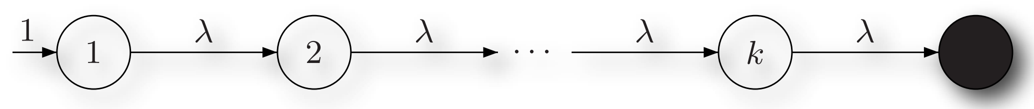
$$\max\{Erl(\lambda_1, k_1), Erl(\lambda_2, k_2), Erl(\lambda_3, k_3)\}$$



Erlang Distributions

Representation:

$Erl(\lambda, k)$



Concurrent Erlangs is their cross product.

Properties:

1. Outgoing rates: for all $\mathcal{A} \subset \{1, 2, \cdots, n\}$ there are $\prod_{i \in \mathcal{A}} k_i$ states with outgoing rates $\sum_{i \in \mathcal{A}} \lambda_i$

2. Paths: for all $\mathcal{A} \subset \{1, 2, \cdots, n\}$ there are at most $(\sum_{i \in \mathcal{A}} k_i) - |\mathcal{A}| + 1$ states with outgoing rates $\sum_{i \in \mathcal{A}} \lambda_i$ each of length $\sum_{i=1}^{n} k_i$

The Minimal of the Maximum

Theorem: The minimal representation of

$$\max\{Erl(\lambda_1, k_1), Erl(\lambda_2, k_2), \cdots, Erl(\lambda_n, k_n)\}$$

is an acyclic phase-type representation having

$$(\sum_{i \in \mathcal{A}} k_i) - |\mathcal{A}| + 1$$

states with outgoing rates

$$\sum_{i \in \mathcal{A}} \lambda_i,$$

for all $\mathcal{A} \subset \{1, 2, \cdots, n\}$

Proof of Existence

Using the core series lemma and properties 2 and 6

2. Paths: for all $\mathcal{A} \subset \{1, 2, \cdots, n\}$ there are at most $(\sum_{i \in \mathcal{A}} k_i) - |\mathcal{A}| + 1$ states with outgoing rates $\sum_{i \in \mathcal{A}} \lambda_i$ each of length $\sum_{i=1}^{n} k_i$

The number of states in the representation:

$$2^{n-1}(\sum_{i=1}^{n} k_i - 1) + 2^n - 1$$

Reflections and Future

- Refining the Cumani's Basic Series
- Maximum of Erlangs as Composition of Erlangs
- Exponential blow-up is inevitable
- Future: generalization